Supercuspidals for GL(2).

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1 Set up

We study smooth representations, the stabilizers are open. F is a non-archimedean local field of residue characteristic p. $\mathcal{O}$ denotes the ring of integers in F and $\mathfrak{p}$ is the maximal ideal. G is the group $\mathrm{GL}_2(F)$ and $K = \mathrm{GL}_2(\mathcal{O})$. B is the group of upper triangular matrices in G and T is the group of diagonal matrices in G. Fix an additive character $\tau\colon F \to \mathbb{C}$. So τ is automatically unitary, i.e. if $x \in F$ then $|\tau(x)| = 1$. If X is a topological space, write $\mathrm{Fun}(X, \mathbb{C})$ for the space of continuous functions $f\colon X \to \mathbb{C}$. So $\mathrm{Fun}(F, \mathbb{C})$ (resp. $\mathrm{Fun}(F^\times, \mathbb{C})$) are the locally constant functions on F (resp. $F^\times$). If *space of Schwartz funcitons* $\mathcal{S}(F)$ (resp. $\mathcal{S}(F^\times)$) is the space of compactly supported functions in $\mathrm{Fun}(F, \mathbb{C})$ (resp. $\mathrm{Fun}(F^\times, \mathbb{C})$). The multiplicative Haar measure $\mathrm{d}^\times x$ is normalized so that the units $\mathcal{O}^\times$ in $\mathcal{O}$ have volume 1. We let $|\cdot|\colon F \to \mathbb{R}_{\geq 0}$ denote the normalized absolute value. If $\pi \in F$ is a uniformiser then $|\pi| = 1/q$ where q denotes the size of the residue field and $\mathrm{d}(cx) = |c|\mathrm{d}x$ for all $c \in F^\times$.

2 A First Look at Principal Series

Let B denote the set of upper triangular matrices in G. Let $\delta\colon B \to \mathbb{C}^\times$ be the *modulus character*

$$\delta\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix} = \left|\frac{t_1}{t_2}\right| \qquad (t_1, t_2 \in F^\times, x \in F).$$

Lemma 2.1. *Suppose* $\mathrm{d}_L b$ *is a left invariant Haar measure on* B*. If* $b_0 \in B$ *then*

$$\mathrm{d}_L(b_0 b b_0^{-1}) = \delta(b_0)\mathrm{d}_L b$$

and $\mathrm{d}_R b := \delta(b)\mathrm{d}_L b$ *is a right invariant Haar measure on* B*.*

Proof. One proves the existence and uniqueness of δ using the existence and uniqueness properties of Haar measure. The fact that $\mathrm{d}_R b = \delta(b)\mathrm{d}_L b$ is right invariant then follows from a formal computation. The measure

$$\mathrm{d}_L\left(\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix}\right) = |t_1|^{-1}\mathrm{d}x\mathrm{d}^\times t_1 \mathrm{d}^\times t_2$$

is left invariant since if $b_0 = \begin{pmatrix} s_1 & y \\ 0 & s_2 \end{pmatrix} \in B$ then

$$\begin{aligned}
\mathrm{d}_L\left(\begin{pmatrix} s_1 & y \\ 0 & s_2 \end{pmatrix}\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix}\right) &= \mathrm{d}_L\left(\begin{pmatrix} s_1t_1 & s_1x+yt_2 \\ 0 & s_2t_2 \end{pmatrix}\right) \\
&= |s_1t_1|^{-1}\mathrm{d}(s_1x+yt_2)\mathrm{d}^\times(s_1t_1)\mathrm{d}^\times(s_2t_2) \\
&= |s_1t_1|^{-1}|t_2|\frac{1}{|t_2|}\mathrm{d}(s_1x+yt_2)\mathrm{d}^\times t_1\mathrm{d}^\times t_2 \\
&= |s_1t_1|^{-1}|t_2|\mathrm{d}\left(\frac{s_1x}{t_2}+y\right)\mathrm{d}^\times t_1\mathrm{d}^\times t_2 \\
&= |s_1t_1|^{-1}|t_2|\mathrm{d}\left(\frac{s_1x}{t_2}\right)\mathrm{d}^\times t_1\mathrm{d}^\times t_2 \\
&= |s_1t_1|^{-1}|s_1|\mathrm{d}x\mathrm{d}^\times t_1\mathrm{d}^\times t_2 \\
&= \mathrm{d}_L\left(\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix}\right).
\end{aligned}$$

So d_L is left invariant. Since $b_0^{-1} = \begin{pmatrix} 1/s_1 & -y/(s_1s_2) \\ 0 & 1/s_2 \end{pmatrix}$,

$$\begin{aligned}
\mathrm{d}_L\left(\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix}b_0^{-1}\right) &= \mathrm{d}_L\left(\begin{pmatrix} t_1/s_1 & x/s_2 - yt_1/(s_1s_2) \\ 0 & t_2/s_2 \end{pmatrix}\right) \\
&= |t_1s_1^{-1}|^{-1}\mathrm{d}\left(\frac{x}{s_2}-\frac{yt_1}{s_1s_2}\right)\mathrm{d}^\times(t_1/s_1)\mathrm{d}^\times(t_2/s_2) \\
&= |s_1||t_1|^{-1}|s_2|^{-1}\mathrm{d}\left(x-\frac{yt_1}{s_2}\right)\mathrm{d}^\times t_1\mathrm{d}^\times t_2 \\
&= \delta(b_0)\mathrm{d}\left(\frac{x}{t_1}-\frac{y}{s_2}\right)\mathrm{d}^\times t_1\mathrm{d}^\times t_2 \\
&= \delta(b_0)\mathrm{d}\left(\frac{x}{t_1}\right)\mathrm{d}^\times t_1\mathrm{d}^\times t_2 \\
&= \delta(b_0)\mathrm{d}_L\left(\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix}\right).
\end{aligned}$$

□

Definition. Let H be a closed subgroup of G, and (σ, W) a smooth representation of H. The *representation* $\mathrm{ind}_H^G\sigma$ *of* G *smoothly induced by* σ is the G-representation (Σ, X) on the vector space

$$X = \left\{ \begin{matrix} \text{locally constant functions} \\ f\colon G \to W \end{matrix} : \begin{matrix} \text{if } h \in H \text{ and } g \in G \text{ then} \\ f(hg) = \sigma(h)f(g) \end{matrix} \right\}$$

with G acting by right translations i.e. if $x, g \in G$ and $f \in X$ then $\Sigma(g)f(x) = f(xg)$.

Let $\mu\colon B \to \mathbb{C}^\times$ be a smooth character. Define $(\rho_\mu, \mathcal{B}_\mu)$ as the representation $\mathrm{Ind}_B^G(\delta^{1/2}\mu)$ so that

$$\mathcal{B}_\mu = \left\{ \begin{matrix} \text{locally constant functions} \\ f\colon G \to \mathbb{C} \end{matrix} : \begin{matrix} \text{if } b \in B \text{ and } g \in G \text{ then} \\ f(bg) = \mu(b)\delta(b)^{1/2}f(g) \end{matrix} \right\}$$

with G acting by right translations i.e. if $x, g \in G$ then $\rho_\mu(g)f(x) = f(xg)$.

Theorem 2.2. ([God18, 1.8]), [Bum97, Theorem 2.6.1]) *Suppose μ is a character of B.*

(i) The representations $(\rho_\mu, \mathcal{B}_\mu)$ is smooth.

(ii) The pairing

$$\langle -,- \rangle \colon B_\mu \times B_{-\mu} \to \mathbb{C}, \qquad \langle \varphi, \psi \rangle = \int_K \varphi(k)\psi(k)\mathrm{d}k, \quad (\varphi \in \mathcal{B}_\mu, \psi \in \mathcal{B}_{-\mu})$$

defines a non-degenerate pairing such that if $\varphi \in \mathcal{B}_\mu$, $\psi \in \mathcal{B}_{-\mu}$ and $g \in G$ then

$$\langle \rho_\mu(g)\varphi, \rho_{-\mu}(g)\psi \rangle = \langle \varphi, \psi \rangle.$$

In particular the representation $(\rho_{-\mu}, \mathcal{B}_{-\mu})$ is naturally isomorphic to the contragredient $(\rho_\mu^\vee, \mathcal{B}_\mu^\vee)$ of $(\rho_\mu, \mathcal{B}_\mu)$.

(iii) If μ is a unitary, i.e. $|\mu(b)| = 1$ for all $b \in B$, then the pairing

$$\langle\langle -,- \rangle\rangle \colon B_\mu \times B_\mu \to \mathbb{C}, \qquad \langle\langle \varphi, \psi \rangle\rangle = \int_K \varphi(k)\overline{\psi(k)}\mathrm{d}k, \quad (\varphi, \psi \in \mathcal{B}_\mu)$$

defines a natural positive definite ρ_μ-invariant Hermitian form on $\mathcal{B}_\mu$ and $\mathcal{B}_\mu$ is unitarilizable, i.e. if $\widehat{\mathcal{B}}_\mu$ denotes the Hilbert space completion of $\mathcal{B}_\mu$ with respect to $\langle\langle -,- \rangle\rangle$ then there is a natural unitary G-action on $\widehat{\mathcal{B}}_\mu$ such that the action map $G \times \widehat{\mathcal{B}}_\mu \to \widehat{\mathcal{B}}_\mu$ is continuous.

Proof. (i) To ease notation write $K = \mathrm{GL}_2(\mathcal{O})$ so that $K \subseteq G$ is a maximal compact subgroup. By the Iwasawa decomposition $G = B \cdot K$. So a function $f \in \mathcal{B}_\mu$ is determined by the restriction $f|_K$. One proves that $f \mapsto f|_K$ defines a K-isomorphism

$$\mathcal{B}_\mu \xrightarrow{\sim} \mathrm{Ind}_{K \cap B}^K(\mu).$$

Since K is compact, the right hand side is spanned by characteristic functions. The smoothness follows.

(ii) Let

$$L(G,P) = \{\varphi \in \mathrm{Fun}(G,\mathbb{C}) \mid \text{if } b \in B \text{ and } g \in G \text{ then } \varphi(bg) = \delta(b)\varphi(g)\}$$

Then G acts on $L(G,B)$ by right translations. If $\varphi \in \mathcal{B}_\mu$ and $\psi \in \mathcal{B}_{-\mu}$ then $\varphi\psi \in L(G,B)$. We prove the form

$$I \colon L(G,B) \to \mathbb{C}, \qquad f \mapsto \int_K f(k)\mathrm{d}k$$

is G invariant. Let $\mathcal{C}_c(G)$ denote the space of compactly supported, locally constant functions on G. For $\phi \in \mathcal{C}_c(G)$ define $\Lambda\phi \colon G \to \mathbb{C}$ by

$$(\Lambda\phi)(g) = \int_B \phi(pg)\mathrm{d}_L p \qquad (g \in G)$$

If $b \in B$ and $g \in G$ then

$$\begin{aligned}(\Lambda\phi)(bg) &= \int_B \phi(pbg)\mathrm{d}_L p = \int_B \phi(pbg)\delta^{-1}(p)\mathrm{d}_R p \\ &= \int_B \phi((pb^{-1})bg)\delta^{-1}(pb^{-1})\mathrm{d}_R(pb^{-1}) \\ &= \int_B \phi(pg)\delta(pb^{-1})\mathrm{d}_R p \\ &= \int_B \phi(pg)\delta^{-1}(pb^{-1})\delta(p)\mathrm{d}_L p \\ &= \delta(b)\int_B \phi(pg)\mathrm{d}_L p \\ &= \delta(b)(\Lambda\phi)(g).\end{aligned}$$

So the functional $\Lambda\phi \in L(G, B)$ and if $h \in H$ then

$$\begin{aligned}I(\Lambda\phi) &= \int_K\int_B \phi(pk)\mathrm{d}_L p\mathrm{d}k = \int_G \phi(g)\mathrm{d}g \\ &= \int_G \phi(gh)\mathrm{d}g = \int_K\int_B \phi(pkh)\mathrm{d}_L p\mathrm{d}k \\ &= I(\Lambda(h\cdot\phi)) \\ &= I(h\cdot\Lambda\phi).\end{aligned}$$

We conclude that I is G-invariant on the elements of $L(G, B)$ which are in the image of the map $\Lambda\colon \mathcal{C}_c(G) \to L(G, B)$. It remains to prove that Λ is surjective. First observe that the restriction of δ to $K \cap B$ is trivial, because $\delta(K \cap B)$ is a compact subgroup of $\mathbb{R}^\times_{>0}$. Thus if $f \in L(G, B)$ then $f|_K$ is constant on the cosets of $K \cap B$. Hence $f|_K$ descends to a continuous function on $(K \cap B)\backslash K$ and we get a linear map $L(G, B) \to \mathrm{Fun}((K \cap B)\backslash K, \mathbb{C})$. Since $G = BK$, an element $f \in L(G, B)$ is unquely determined by $f|_K$. So $L(G, B) \to \mathrm{Fun}((K\cap B)\backslash K, \mathbb{C})$ is injective. The surjectivity of Λ will follow from the proof that the composition

$$\mathcal{C}_c(G) \xrightarrow{\Lambda} L(G, B) \hookrightarrow \mathrm{Fun}((K \cap B)\backslash K, \mathbb{C})$$

is surjective. Let $f \in \mathrm{Fun}((B \cap K)\backslash K, \mathbb{C})$ and

$$\phi_0 = \frac{\mathbf{1}_{B\cap K}}{\mathrm{Vol}(K \cap B)}$$

denote the indicator function $\mathbf{1}_{B\cap K}$ of $B \cap K$ renormalized so that $\int_G \phi_0\mathrm{d}g = 1$. Since $G = PK$, if $p \in P$ and $k \in K$ then the formula $\phi(pk) = \phi_0(p)f(k)$ gives a well defined element $\phi \in \mathcal{C}(G)$ and

$$(\Lambda\phi)(k) = \int_B \phi(pk)\mathrm{d}p = \int_B \phi_0(p)f(k)\mathrm{d}p = f(k)\int_B \phi_0(p)\mathrm{d}p = f(k)$$

So $(-)|_K \circ \Lambda \colon \mathcal{C}(G) \to \mathrm{Fun}((K \cap B)\backslash K, \mathbb{C})$ is surjective. We conclude that Λ is surjective. In summary we have proven,

$$\langle \varphi, \psi \rangle = \int_K \varphi(k)\psi(k)\mathrm{d}k$$

defines a bilinear pairing $\langle -, - \rangle \colon B_\mu \times B_{-\mu} \to \mathbb{C}$ such that if $\varphi \in \mathcal{B}_\mu$, $\psi \in \mathcal{B}_{-\mu}$, and $g \in G$ then

$$\langle \rho_\mu(g)\varphi, \rho_{-\mu}(g)\psi \rangle = \langle \varphi, \psi \rangle.$$

It remains to show that this is non-degenerate. Suppose $\varphi \in \mathcal{B}_\mu$ is non-zero. Since $K \cap B$ is profinite, $\mu|_K$ is unitary and $\overline{\mu(b)} = \mu(b)^{-1}$ for all $b \in K \cap B$. So if $k \in K$ and $b \in B \cap K$ then

$$\overline{\varphi(bk)} = \overline{\mu(b)\varphi(k)} = \mu(b)^{-1}\overline{\varphi(k)}$$

So $\overline{\varphi}|_K$ is an element in $I^K_{K\cap B}(-\mu)$ and $\overline{\varphi}|_K$ extends uniquely to and element in $\varphi^* \in \mathcal{B}_\mu$. Since

$$\langle \varphi, \varphi^* \rangle = \int_K \varphi(k)\overline{\varphi(k)}\mathrm{d}k = \int_K |\varphi(k)|^2 \mathrm{d}k \neq 0,$$

the pairing is non-degenerate.

(iii) Suppose μ is unitary, then $\overline{\varphi} \in \mathcal{B}_{-\mu}$ for all $\varphi \in \mathcal{B}_\mu$. So in the notation of (ii), if $\varphi, \psi \in \mathcal{B}_\mu$ then

$$\langle\langle \varphi, \psi \rangle\rangle = \int_K \varphi(k)\overline{\psi(k)}\mathrm{d}k = \langle \varphi, \overline{\psi} \rangle.$$

Therefore $\langle\langle -, - \rangle\rangle$ is G-equivariant by the result of (ii). This proves that $\langle\langle -, - \rangle\rangle$ defines a ρ_μ-invariant Hermitian form on $\mathcal{B}_\mu$. So the G action on $\mathcal{B}_\mu$ extends to a G-action on the Hilbert space completion $\widehat{\mathcal{B}}_\mu$ of $\mathcal{B}_\mu$. To show that the G action on $\widehat{\mathcal{B}}_\mu$ is unitary, one must prove that the action map

$$G \times \widehat{\mathcal{B}}_\mu \to \widehat{\mathcal{B}}_\mu$$

is continuous when $\widehat{\mathcal{B}}_\mu$ is topologized via the Hilbert space topology. The details can be found in [Bum97, 2.6].

□

3 Supercuspidals

In this section π is an irreducible admissible representation of G on a vector space V.

3.1 Kirillov Models

Definition. A *Kirillov model for* π is an admissible representation (ρ, W) on a vector subspace $W \subseteq \mathrm{Fun}(F, \mathbb{C})$ such that:

(i) $\pi \simeq \rho$ as representations of G.

(ii) If $a, x \in F^\times$, $b \in F$ and $\xi' \in W$, then

$$\rho \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \xi'(x) = \tau_F(bx)\xi'(ax).$$

Let

$$V_0 = \left\{ \xi \in V \mid \text{there exists } n \in \mathbb{Z} \text{ such that } \int_{\mathfrak{p}^{-n}} \overline{\tau(x)} \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \xi \mathrm{d}x = 0 \right\}.$$

The subspace $V_0 \subseteq V$ is G-stable and $\dim(V/V_0) = 1$ [God18, 1.2 Lemma 6]. Fix an identification $V/V_0 = \mathbb{C}$ and let $K(\pi)$ denote the image of V under the map

$$V \to \mathrm{Fun}(F, \mathbb{C}), \quad \xi \mapsto \xi' \colon \xi'(t) := \pi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \xi + V_0 \qquad (t \in F^\times).$$

This map is injective [God18, §1.2], so by transport of structure $K(\pi)$ gives a G-representation $(\pi_K, K(\pi))$ such that $(\pi_K, K(\pi)$ is isomorphic to (π, V).

Theorem 3.1. ([God18, 1.2 Theorem 1])
(i) The representation $(\pi_K, K(\pi))$ is the unique Kirillov model for π.
(ii) $\mathcal{S}(F^\times) \subseteq K(\pi)$ as a vector subspace with finite codimension.
(iii) If

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then $K(\pi) = \mathcal{S}(F^\times) + \pi(w)\mathcal{S}(F^\times)$.

3.2 Invariant Duality

Theorem 3.2. ([God18, 1.2 Theorem 2]) *Write ω_π for the central character of π.*
(i) If $\pi^\vee$ denotes the contragredient of π then $\pi^\vee \simeq \omega_\pi^{-1} \otimes \pi$.
(ii) The vector space underlying the Kirillov model of $\pi^\vee$ is

$$K(\pi^\vee) = \{\xi^\vee \colon x \mapsto \omega_\pi(x)^{-1}\xi(x) \mid \xi \in K(\pi)\}.$$

(iii) If $\xi \in K(\pi) = \mathcal{S}(F^\times) + \pi(w)\mathcal{S}(F^\times)$, $\eta \in K(\pi^\vee)$ and $\xi_1, \xi_2 \in \mathcal{S}(F^\times)$ satisfy $\xi = \xi_1 + \pi(w)\xi_2$ then

$$\langle \xi, \eta \rangle = \int \xi_1(x)\eta(-x)\mathrm{d}^\times x + \int \xi_2(x) \cdot \pi^\vee(w)\eta(-x)\mathrm{d}^\times x$$

defines a G-invariant bilinear form between $K(\pi)$ and $K(\pi^\vee)$.

3.3 Supercuspidals Representations

Proposition 3.3. *[God18, 1.8] If $K(\pi) \supsetneq \mathcal{S}(F^\times)$ then there exists μ such that π is isomorphic to a subrepresentation of ρ_μ*

Proof. Suppose $K(\pi) \supsetneq \mathcal{S}(F^\times)$. The Borel subgroup $B \subseteq G$ decomposes as a product ZM where

$$M = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}$$

and Z is the center of G. Since π is irreducible, Z acts on $K(\pi)$ via the central character of π. If $m = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, $x \in F$ and $\xi \in K(\pi)$ then $m\xi(x) = \tau(ax)\xi(bx)$. So the action of B on $K(\pi)$ preserves the subspace $K(\pi) \subseteq \mathcal{S}(F^\times)$. So B acts on the non-trivial finite dimensional quotient $K(\pi)/\mathcal{S}(F^\times)$.

Since τ has a non-trivial conductor, if $\xi \in \mathcal{S}(F)$ and $b \in F$ then $x \mapsto (\tau(bx) - 1)\xi(x)$ is trivial on a ball of non-zero radius around 0 . Hence the subgroup

$$U := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$$

operates trivially on $K(\pi)/\mathcal{S}(F^\times)$. So B operates on $K(\pi)/\mathcal{S}(F^\times)$ via the torus

$$T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \simeq B/U.$$

So B operates on the finite dimensional space $K(\pi)/\mathcal{S}(F^\times)$ by pairwise commuting linear operators. Hence the elements in B have a common eigenvector which spans a one-dimensional B-stable subspace W in $K(\pi)/\mathcal{S}(F^\times)$. Hence there is a well defined projection $L\colon K(\pi) \to W = \mathbb{C}$ and a smooth character μ of B such that if $b \in B$ and $\xi \in K(\pi)$ then

$$L(\pi(b)\xi) = \mu(b)\delta(b)^{1/2}L(\xi).$$

The mapping $\xi \mapsto \varphi_\xi\colon \varphi_\xi(g) = L(\pi(g)\xi)$ is then an isomorphism of π onto a G-stable submodule of B_μ. $\square$

Definition. A *supercuspidal representation of* G is an irreducible smooth representation (π, V) such that if $v \in V$ and $v^\vee \in V^\vee$ then the matrix coefficient $\gamma_{v\otimes v^\vee}$ is compactly supported modulo Z.

Theorem 3.4. *[God18, 1.7] Let π be an irreducible admissible representation of G on a vector space V. The following conditions are equivalent.*

(i) $K(\pi) = \mathcal{S}(F^\times)$.

(ii) There exists $n \in \mathbb{Z}_{\geq 0}$ such that if $\xi \in K(\pi)$ then the function

$$y \mapsto \int_{\mathfrak{p}^{-n}} \left(\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \xi \right)(y)\mathrm{d}x$$

is identically zero.

(iii) π is supercuspidal.

Proof. (i) $\iff$ (ii) Let $\mathfrak{p}^{-d}$ denote the largest fractional ideal of F on which τ is trivial. If $y \in F$, $n \in \mathbb{Z}$ and $\xi \in K(\pi)$ then

$$\int_{\mathfrak{p}^{-n}} \tau(xy)\mathrm{d}x \neq 0$$

if and only if $y \in \mathfrak{p}^{n-d}$. So if $n \in \mathbb{Z}$ and $\xi \in K(\pi)$ then

$$y \mapsto \int_{\mathfrak{p}^{-n}} \left(\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \xi \right)(y)\mathrm{d}x = \xi(y) \int_{\mathfrak{p}^{-n}} \tau(xy)\mathrm{d}x$$

is identically zero if and only if $\mathfrak{p}^{n-d} \subseteq F - \mathrm{supp}(\xi)$. There exists n such that $\mathfrak{p}^{n-d} \subseteq F - \mathrm{supp}(\xi)$ if and only if $\xi \in \mathcal{S}(F^\times)$. So (i) and (ii) are equivalent.

(i) $\implies$ (iii) Assume π is supercuspidal. By Theorem 3.2, $K(\pi^\vee)$ is obtained from π by multiplying the functions $\xi \in K(\pi) = \mathcal{S}(F^\times)$ by the locally constant central character ω_π. Hence $K(\pi^\vee)$ is supercuspidal. Theorem 3.2 says that the invariant duality between $K(\pi)$ and $K(\pi^\vee)$ is given by the bilinear form

$$\langle \xi, \eta \rangle = \int_{F^\times} \xi(x)\eta(-x)\mathrm{d}^\times x, \qquad (\xi, \eta \in \mathcal{S}(F^\times)).$$

Fix $\xi \in K(\pi)$ and $\eta \in K(\pi^\vee)$. Since $G = KTK$ and K is compact, it suffices to show that the function

$$T \mapsto \mathbb{C}, \qquad t \mapsto \langle \pi(t)\xi, \eta \rangle$$

has compact support modulo Z. Since $T = Z \cdot \left\{ \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \right\}$, this is equivalent to showing

$$F^\times \to \mathbb{C}, \qquad t \mapsto \langle \pi \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \xi, \eta \rangle = \int_{F^\times} \xi(tx)\eta(-x)\mathrm{d}^\times x$$

has compact support. This is true because $\xi, \eta \in \mathcal{S}(F^\times)$.

(i) $\Longleftarrow$ (iii) Assume (π, V) satisfies: if $\xi \in V$ and $\eta \in V^\vee$ then $g \mapsto \langle \pi(g)\xi, \eta \rangle$ has compact support modulo the center Z of G. We show that π is supercuspidal. We can assume $V = K(\pi)$ and $V^\vee = K(\pi^\vee)$. Then $\mathcal{S}(F^\times) \subseteq V$ and we let $\xi \in \mathcal{S}(F^\times)$ and $\eta \in K(\pi^\vee)$. Our description of the invariant duality (Theorem 3.2) between $K(\pi)$ and $K(\pi^\vee)$ yields

$$\langle \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \xi, \eta \rangle = \int \xi(tx)\eta(-x)\mathrm{d}^\times x. \tag{3.1}$$

As we discussed in the previous implication, the matrix coefficients having compact support implies (3.1) is compactly support. Since $\xi \in \mathcal{S}(F^\times)$ was arbitrary, it follows that $\eta \in \mathcal{S}(F^\times)$. So $K(\pi^\vee) = \mathcal{S}(F^\times)$ applying Theorem 3.2 we conclude $\mathcal{S}(F^\times) = K(\pi)$. So π is supercuspidal.

□

4 Simple Supercuspidals

We explain the construction of a **simple** class of supercuspidal representations for $\mathrm{GL}_2(F)$. These representation were discovered by Mark Reeder [GR10, §8]. His construction can be applied to give very simple examples of irreducible supercuspidals quite generally (i.e for all simple, split, simply connected groups, Sp_{2n}, G_2, E_8, etc.). The case of $\mathrm{GL}(n)$ requires a minor modification to [GR10, §9] and is worked out, for example, in [KL15].

Fix a uniformizer $\varpi \in \mathcal{O}$ and a tamely ramified character $\omega \colon F^\times \to \mathbb{C}^\times$, i.e. a character ω such that $\omega(1 + \varpi\mathcal{O}) = 1$. The construction will yield $2(q-1)$ non-isomorphic supercuspidal representation of $\mathrm{GL}_2(F)$ with central character ω.

The *pro-p Iwahori in* $\mathrm{GL}(2)$ is

$$I^+ = \left\{ \begin{pmatrix} 1 + \varpi a_1 & b \\ \varpi c & 1 + \varpi a_2 \end{pmatrix} \in \mathrm{GL}_2(F) \colon a_1, a_2, b, c \in \mathcal{O} \right\}.$$

So I^+ is the preimage of the upper triangular unipotent matrices in $\mathrm{GL}_2(\mathcal{O}/\varpi)$ under the reduction map $\mathrm{GL}_2(\mathcal{O}) \to \mathrm{GL}_2(\mathcal{O}/\varpi)$. Let $\zeta \in \mu_{q-1}(F)$ be a $(q-1)$st root of unity and let χ_ζ be the *affine generic character*[1]

$$\chi_\zeta\colon ZI^+ \to \mathbb{C}^\times, \qquad \chi_\zeta\left(z\begin{pmatrix}1+\varpi a_1 & b\\ \varpi c & 1+\varpi a_2\end{pmatrix}\right) = \omega(z)\tau\left(b+\frac{c}{\zeta}\right)$$

where $z \in F^\times$, $\begin{pmatrix}1+\varpi a_1 & b\\ \varpi c & 1+\varpi a_2\end{pmatrix} \in I^+$, and $\tau\colon F \to \mathbb{C}^\times$ is a fixed additive character with conductor $\varpi\mathcal{O}$. Since ω is tamely ramified and

$$I^+ \cap F^\times = 1+\varpi\mathcal{O},$$

the character $\chi_\zeta\colon ZI^+ \to \mathbb{C}^\times$ is well defined. Write

$$\beta_\zeta = \begin{pmatrix}0 & 1\\ \zeta\varpi & 0\end{pmatrix} \quad \text{so that} \quad \beta_\zeta^2 = \zeta\varpi.$$

Then β_ζ normalizes ZI^+ and if $k \in ZI^+$ then

$$\chi_\zeta(\beta_\zeta k \beta_\zeta^{-1}) = \chi_\zeta(k).$$

Hence if $\xi = \omega(\zeta\varpi)^{\frac{1}{2}}$ is square root of $\omega(\zeta\varpi)$ in $\mathbb{C}^\times$ then

$$\chi_\zeta^\xi\colon \langle\beta_\zeta\rangle ZI^+ \to \mathbb{C}^\times, \quad \beta_\zeta^i k \mapsto \xi^i\chi_\zeta(k), \qquad (i\in\mathbb{Z}, k \in ZI^+)$$

is an extension of χ_ζ to $\langle\beta_\zeta\rangle ZI^+$, the product of ZI^+ with the cyclic subgroup $\langle\beta_\zeta\rangle$.

Theorem 4.1. ([GR10, Proposition 9.3]) *The compact induction*

$$\pi_{\zeta,\xi} = \mathrm{ind}_{\langle\beta_\zeta\rangle ZI^+}^G \chi_\zeta^\xi \qquad \textit{is irreducible and supercuspidal}$$

with central character ω. There are $2(q-1)$ choices for the pair $(\zeta,\xi) \in \mu_{q-1}(F)\times\mathbb{C}^\times$ such that $\xi^2 = \omega(\zeta\varpi)$. If (ζ_1,ξ_1) and (ζ_2,ξ_2) are two such choices then $\pi_{\zeta_1,\xi_1} \simeq \pi_{\zeta_2,\xi_1}$ if and only if $\zeta_1=\zeta_2$ and $\xi_1=\xi_2$.

As a $\mathbb{C}$-vector space

$$\mathrm{ind}_{\langle\beta_\zeta\rangle ZI^+}^G \chi_\zeta^\xi = \left\{ f \in \mathrm{Fun}(G,\mathbb{C})\colon \begin{array}{c}\text{The support of } f \text{ is compact modulo } Z \text{ and if}\\ k \in \langle\beta_\zeta\rangle ZI^+ \text{ and } g \in G \text{ then } f(k\cdot g) = \chi_\zeta^\xi(k)f(g)\end{array} \right\}.$$

The group G acts on c-$\mathrm{ind}_{\langle\beta_\zeta\rangle ZI^+}^G \chi_\zeta^\xi$ by right translations, i.e if $f \in \mathrm{ind}_{\langle\beta_\zeta\rangle ZI^+}^G \chi_\zeta^\xi$ and $x, g \in G$ then $(g\cdot f)(x) = f(xg)$.

[1] $\chi_\zeta|_{I^+}$ is non-trivial on a root subgroup U_ψ if and only if ψ is a simple affine root. This property of χ_ζ is the one which generalizes to other groups.

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